2023 AP Calculus AB Free Response Questions
Section II, Part A (30 minutes)
# of questions: 2

A graphing calculator may be used for this part

t (seconds)	0	60	90	120	135	150
$f(t)$ (gallons per second)	0	0.1	0.15	0.1	0.05	0

1. A customer at a gas station is pumping gasoline into a gas tank. The rate of flow of gasoline is modeled by a differentiable function f, where $f(t)$ is measured in gallons per second and t is measured in seconds since pumping began. Selected values of $f(t)$ are given in the table above.
   (a) Using correct units, interpret the meaning of $\int_{60}^{135} f(t)\,dt$ in the context of the problem. Use a right Riemann sum with the three subintervals [60,90], [90,120], and [120,135] to approximate the value of $\int_{60}^{135} f(t)\,dt$.
   (b) Must there exist a value of c, for $60 < c < 120$, such that $f'(c) = 0$? Justify your answer.
   (c) The rate of flow of gasoline, in gallons per second, can also be modeled by $g(t) = \left(\frac{t}{500}\cos\left(\left(\frac{t}{120}\right)^2\right)\right)$ for $0 \le t \le 150$. Using this model, find the average rate of flow of gasoline over the time interval $0 \le t \le 150$. Show the setup for your calculations.
   (d) Using the model g defined in part (c), find the value of $g'(140)$. Interpret the meaning of your answer in the context of the problem.

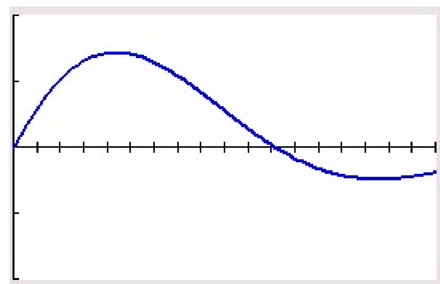
(a) The amount of gas pumped into the tank, in gallons, for time interval from 60 seconds to 135 seconds.
$\int_{60}^{135} f(t)\,dt = 30f(90) + 30f(120) + 15f(135) = 30(0.15) + 30(0.1) + 15(0.05)$
$\int_{60}^{135} f(t)\,dt =$ 8.25 gallons

(b) Yes. From $60 \le t \le 90$, f is increasing. Therefore $f'(t) > 0$. From $90 \le t \le 135$, f is decreasing, so $f'(t) < 0$. Since the function is differentiable, it is continuous, therefore by the IVT, there exists a c, $60 < c < 120$, such that $f'(c) = 0$.

(c) $g_{avg}(t) = \frac{1}{150-0}\int_0^{15} g(t)dt = 0.095996 \approx$ 0.096 gal/sec
(d) $g'(140) = -0.00491 \approx -0.005$ gal/sec^2. The rate of change of the flow of gas at $t = 140$ seconds (which is decreasing).

2. Stephen swims back and forth along a straight path in a 50-meter-long pool for 90 seconds. Stephen's velocity is modeled by $v(t) = 2.38e^{-0.02t} \sin\left(\frac{\pi}{56}t\right)$, where t is measured in seconds and $v(t)$ is measured in meters per second.
   (a) Find all times t in the interval $0 < t < 90$ at which Stephen changes direction. Give a reason for your answer.
   (b) Find Stephen's acceleration at time $t = 60$ seconds. Show the setup for your calculations and indicate units of measure. Is Stephen speeding up or slowing down at time $t = 60$ seconds? Give a reason for your answer.
   (c) Find the distance between Stephen's position at time $t = 20$ seconds and his position at time $t = 80$ seconds. Show the setup for your calculations.
   (d) Find the total distance Stephen swims over the time interval $0 \le t \le 90$ seconds. Show the setup for your calculations.

(a) Stephen changes direction when there is a sign change of $v(t)$ over the points where $v(t) = 0$:
To the right is the graph of $v(t)$. Stephen changes direction when the curve crosses the t–axis (or x-axis). This is when the velocity changes from a positive value to a negative value. This occurs at $t = 56$ sec.



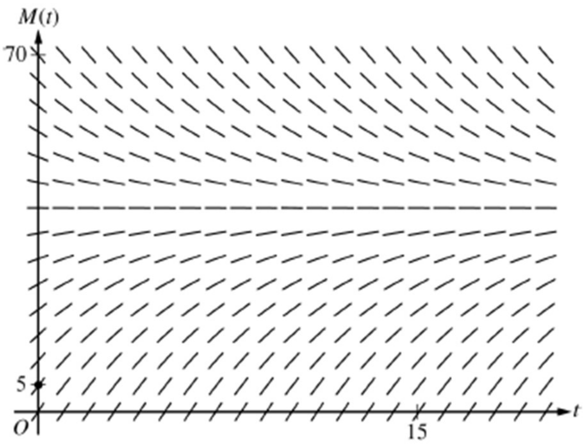
(b) $a(60) = v'(60) = -0.036$ m/sec^2.
Since $v(60) = -0.16$ has the same sign as $a(60)$, then Stephen is speeding up.

(c) Distance of position = net distance = $\int_{20}^{80} v(t)dt = 23.233$ *or* 23.234 meters

(d) $s_{tot} = \int_0^{90} |v(t)|dt = 62.164$ meters.

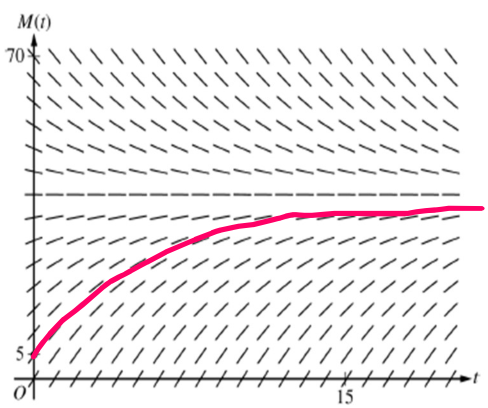
Section II, Part B (1 hour)
# of questions: 4
A graphing calculator may NOT be used for this part.

3. A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius (°C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation
$\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.
   (a) A slope field for the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ is shown to the right. Sketch the solution curve through the point (0,5).



   (b) Using the tangent line to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time t=2 minutes.
   (c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.
   (d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

(a) See the graph to the right.



(b) At $t = 0$, $M(0) = 5$, therefore $M(2) = M(0) + M'(0)t$

At $t = 0$, $\frac{dM}{dt} = \frac{1}{4}(40 - M(0)) = \frac{1}{4}(40 - 5) = \frac{35}{4}$

$M(t) = 5 + \frac{35}{4}t \rightarrow M(2) = 5 + \frac{70}{4} = 22.5°\text{C}$

(c) $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} \rightarrow \frac{d^2M}{dt^2} = -\frac{1}{16}(40 - M)$

at $t = 2$, $\frac{d^2M}{dt^2} = -\frac{1}{16}(35) = -\frac{35}{16} < 0$

Since it is negative, it is concave down, the tangent would be above the curve making it an overestimate.

(d) $\frac{dM}{40-M} = \frac{1}{4}dt$

$$\int \frac{dM}{40 - M} = \int \frac{1}{4}dt$$

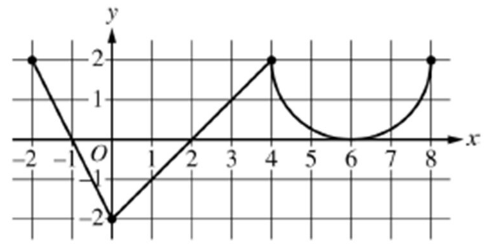
$$-\ln|40 - M| = \frac{1}{4}t + C$$

At $t = 0$, $M = 5$:

$$-\ln 35 = C \rightarrow -\ln|40 - M| = \frac{1}{4}t - \ln 35 \rightarrow \ln|40 - M| = -\frac{1}{4}t + \ln 35$$

$$40 - M = \pm e^{-0.25t + \ln 35} \rightarrow M = 40 \pm e^{\ln 35}e^{-0.25t} \rightarrow M = 40 \pm 35e^{-0.25}$$

Since $M(0) = 5 \rightarrow M = 40 - 35e^{-0.25t}$



Graph of f'

4. A function f is defined on the closed interval $[-2,8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure above.
   (a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.
   (b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.
   (c) Find the value of $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6}$, or show that it does not exist. Justify your answer.
   (d) Find the absolute minimum value of f on the closed interval $[-2,8]$. Justify your answer.

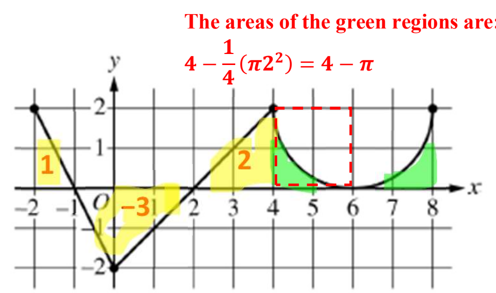
(a) At $x = 6$, the graph of f' is not crossing the x-axis, therefore there is no sign change of f'. Because there is no sign change, the answer is NEITHER.

(b) f is concave down when $f'' < 0$ or when f' is decreasing. This occurs on the intervals: $(-2,0)$ and $(4,6)$.

(c) $\lim_{x\to 2} \frac{6f(x)-3x}{x^2-5x+6} = \frac{6(1)-3(2)}{4-10+6} = \frac{0}{0}$

By L'Hopital's Rule:

$$\lim_{x\to 2} \frac{6f'(x)-3}{2x-5} = \frac{6(0)-3}{2(2)-5} = \frac{-3}{-1} = 3$$



Graph of f'

(d) Areas of regions are marked in RED
Absolute min will occur at a relative min or the endpoints……
The rel. min occurs when f' crosses the x-axis from negative to positive (at $x = 2$). Testing $x = 2$ and the endpoints:

$$f(8) = 1 + 2 + (4-\pi) + (4-\pi) = 11 - 2\pi$$
$$f(2) = 1 \quad \text{<= absolute minimum value is 1}$$
$$f(-2) = 1 - (-3) - 1 = 3$$

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

5. The functions f and g are twice differentiable. The table above gives values of the functions and their first derivatives at selected values of x.
   (a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.
   (b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.
   (c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)dt$. Find $m(2)$. Show the work that leads to your answer.
   (d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$?

(a) $h'(x) = f'(g(x)) \cdot g'(x)$
$h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 1$

(b) $k''(x) = (f(x))^2 \cdot g'(x) + g(x) \cdot 2f(x)f'(x) = (f(x))^2 \cdot g'(x) + 2f(x)g(x)f'(x)$
$k''(4) = (f(4))^2 \cdot g'(4) + 2f(4)g(4)f'(4) = 16(2) + 2(4)(-3)(3) = 32 - 72 = -40$
Since $k''(4) = -40 < 0$, then k is concave down.

(c) $m(2) = 5(2)^3 + \int_0^2 f'(t)dt = 40 + f(t)\Big|_0^2 = 40 + [f(2) - f(0)] = 40 + (-3) = 37$

(d) $m'(x) = \frac{d}{dx}\left[5x^3 + \int_0^x f'(t)dt\right] = 15x^2 + f'(x)$
$m'(2) = 40 + f'(2) = 60 - 8 = 52 > 0$ $\therefore$increasing

6. Consider the curve given by the equation $6xy = 2 + y^3$.

(a) Show that $\frac{dy}{dx} = \frac{2y}{y^2-2x}$.

(b) Find the coordinates of a point on the curve at which the line tangent to the curve is horizontal or explain why no such point exists.

(c) Find the coordinates of a point on the curve at which the line tangent to the curve is vertical or explain why no such point exists.

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{2}, -2\right)$, its horizontal position is increasing at a rate of $\frac{dx}{dt} = \frac{2}{3}$ units per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

(a) $6xy = 2 + y^3 \rightarrow \; 6x\frac{dy}{dx} + y(6) = 3y^2\frac{dy}{dx} \;\rightarrow\; 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y \rightarrow \frac{dy}{dx}(3y^2 - 6x) = 6y$

$$\frac{dy}{dx} = \frac{6y}{3y^2 - 6x} = \frac{2y}{y^2 - 2x}$$

(b) There will be a horizontal tangent when $\frac{dy}{dx} = 0$ or when $6y = 0 \rightarrow \; y = 0$

At $y = 0$: $6x(0) = 2 + 0^3 \rightarrow \; 0 = 2$ Never $\therefore$ No horizontal tangents.

(c) There will be vertical tangent when $\frac{dy}{dx}$ is undefined or when $y^2 - 2x = 0 \rightarrow x = \frac{1}{2}y^2$

$6\left(\frac{1}{2}y^2\right)y = 2 + y^3 \rightarrow 3y^3 = 2 + y^3 \rightarrow \; 2y^3 = 2 \;\rightarrow\; y^3 = 1 \rightarrow \; y = 1$

At $y = 1$: $x = \frac{1}{2}(1)^2 = \frac{1}{2}$ So there is a vertical tangent through $\left(\frac{1}{2}, 1\right)$

(d) $6xy = 2 + y^3$ By implicit differentiation with respect to t:

$$6x\frac{dy}{dt} + 6y\frac{dx}{dt} = 3y^2\frac{dy}{dt} \rightarrow \frac{dy}{dt}(3y^2 - 6x) = 6y\frac{dx}{dt} \rightarrow \; \frac{dy}{dt} = \frac{6y\frac{dx}{dt}}{3y^2 - 6x}$$

At $\left(\frac{1}{2}, -2\right)$ *and* $\frac{dx}{dt} = \frac{2}{3}$: $\frac{dy}{dt} = \frac{6(-2)\left(\frac{2}{3}\right)}{3(4)-3} = -\frac{8}{9}$ units/sec